10.3-10.6: Convergence Tests and Bounds

Definitions.

1. The p-series is of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for any $p \in \mathbb{R}$.

2. A series $\sum a_n$ converges absolutely (or is absolutely convergent) if the corresponding series of absolute values, $\sum |a_n|$, converges.

3. A series whose terms alternate between positive and negative values is called an alternating series.

4. A series $\sum a_n$ is converges conditionally (or is conditionally convergent) if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Convergence Tests.

1. **The nth-Term Test:** If it is not true that $a_n \to 0$, then the series diverges.

2. **The Monotonic Sequence Theorem:** A series $\sum a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

3. **The Integral Test:** Let $\{a_n\}$ be a sequence of positive terms. Suppose $a_n = f(n)$, where f is a continuous, positive, decreasing function of x such that $\exists N \in \mathbb{N}$ such that $\forall x \geq N$. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.

4. **The Comparison Test:** Let $\sum a_n$, $\sum b_n$ and $\sum c_n$ be series with non-negative terms. Suppose that for some $N \in \mathbb{N}$

$$a_n \leq b_n \leq c_n, \quad \forall n \geq N.$$

   (a) If $\sum c_n$ converges, then $\sum b_n$ also converges.

   (b) If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

5. **Limit Comparison Test:** Suppose that $\exists N \in \mathbb{N}$ such that $a_n > 0$ and $b_n > 0 \ \forall n \geq N$.

   (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

   (b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

   (c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

6. **The Absolute Convergence Test:** If $\sum |a_n|$ converges, then $\sum a_n$ converges.

7. **The Ratio Test:** Let $\sum a_n$ be a series and suppose that

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n} = r.$$

   (a) If $r < 1$, then the series converges absolutely.
   (b) If $r > 1$, then the series diverges.
   (c) If $r = 1$, the test is inconclusive.

8. **The Root Test:** Let $\sum a_n$ be a series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r.$$

   (a) If $r < 1$, then the series converges absolutely.
   (b) If $r > 1$, then the series diverges.
   (c) If $r = 1$, the test is inconclusive.

9. **The Alternating Series Test:** The series $\sum_{n=1}^{\infty}(-1)^n u_n$ converges if the u_n's are all positive, the u_n's are (eventually) decreasing and $u_n \to 0$.

Which Tests to Use.

1. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.

2. p**-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.

3. **Series with Nonnegative Terms:** Try the Integral Test or try comparing to a known series with the Comparison Test or the Limit Comparison Test. You can also try the Ratio or Root Test.

4. **Series with Some Negative Terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another test from above? Remember that absolute convergence implies convergence.

5. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

Bounds for Convergence Tests.

1. **The Integral Test:** Let $\{a_n\}$, f and $N \in \mathbb{N}$ satisfy the conditions of the integral test. Then the remainder $R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

2. **The Alternating Series Test:** Let $\sum_{n=1}^{\infty}(-1)^n u_n$ satisfy the conditions of the alternating series test. Then the remainder R_n, as above, has the same sign as a_{n+1} and satisfies the inequality

$$|R_n| \leq |a_{n+1}|.$$